Exercise CW-151

Deriving a Cobb-Douglas expenditure function

The Economic Skills Project

1 Problem

Problem

A household consumes two goods, X and Y, and has Cobb-Douglas preferences. Its utility function and demand equations are:

$$U = X^{0.25}Y^{0.75}, \;\; X = \frac{0.25M}{P_X}, \;\; Y = \frac{0.75M}{P_Y}$$

Derive the expenditure function for the household.

2 Answer

Answer

Here are two ways the expenditure function can be written; either is correct:

$$M = U \cdot \left(\frac{P_X}{0.25}\right)^{0.25} \left(\frac{P_Y}{0.75}\right)^{0.75}$$

or

$$M = \frac{U}{\left(\frac{0.25}{P_X}\right)^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75}}$$

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3 Method

Solution method

Here's one approach:

1. Substitute the demand equations into the utility function.
2. Collect terms in M.
3. Solve the resulting equation for M.

4 Solution

4.1 Step 1

Substitute the demand equations into the utility function

Here's the result:

$$U = \left(\frac{0.25M}{P_X}\right)^{0.25} \left(\frac{0.75M}{P_Y}\right)^{0.75}$$

4.2 Step 2

Factor out M

The first step is to use the rule of exponents that says $(X \cdot Y)^a = X^a Y^a$ to extract M from the terms in parentheses:

$$U = \left(\frac{0.25}{P_X}\right)^{0.25} M^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75} M^{0.75}$$

Then collect the M terms together:

$$U = \left(\frac{0.25}{P_X}\right)^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75} M^{0.25} M^{0.75}$$

Continued...

Factor out M

Now use the rule of exponents that says $X^a X^b = X^{a+b}$:

$$U = \left(\frac{0.25}{P_X}\right)^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75} M$$

Formally, this is known as the household's *indirect* utility function because it reports utility as a function of prices and income: $U(P_X, P_Y, M)$. The direct utility function was the one at the start of the problem that reports utility as a function the goods the household buys: $U(X, Y)$.

4.3 Step 3

Solve for M

Dividing both sides by the terms involving prices:

$$M = \frac{U}{\left(\frac{0.25}{P_X}\right)^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75}}$$

For a more compact version, multiply the numerator and denominator by the reciprocal of the denominator:

$$M = \frac{U}{\left(\frac{0.25}{P_X}\right)^{0.25} \left(\frac{0.75}{P_Y}\right)^{0.75}} \cdot \frac{\left(\frac{P_X}{0.25}\right)^{0.25} \left(\frac{P_Y}{0.75}\right)^{0.75}}{\left(\frac{P_X}{0.25}\right)^{0.25} \left(\frac{P_Y}{0.75}\right)^{0.75}}$$

Continued...

Solve for M

All the terms in the denominator now multiply out to 1 giving:

$$M = U \cdot \left(\frac{P_X}{0.25}\right)^{0.25} \left(\frac{P_Y}{0.75}\right)^{0.75}$$

Done!