Quick Reference Guide to Common Functional Forms

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Contents

1 Consumption

1.1 Leontief

Zero elasticity of substitution; own-price elasticities are less than one; cross-price elasticities are negative; homothetic.

Utility:

$$u = \min \left\{ \frac{x_i}{\alpha_i} \right\} \tag{1}$$

Indirect Utility:

$$v = m \left(\sum_{i=1}^{N} \alpha_i p_i \right)^{-1} \tag{2}$$

Expenditure Function:

$$e = u \sum_{i=1}^{N} \alpha_i p_i \tag{3}$$

Price Index:

$$p_u = \sum_{i=1}^{N} \alpha_i p_i \tag{4}$$

Typical Demand Equation:

$$x_i = \alpha_i \frac{m}{p_u} \tag{5}$$

Income Elasticity:

$$\eta_m = 1 \tag{6}$$

Expenditure Share:

$$s_i = \alpha_i \frac{p_i}{p_u} \tag{7}$$

1.2 Cobb-Douglas

Unitary elasticity of substitution; own-price elasticities are equal to one; cross-price elasticities are zero; homothetic.

Utility:

$$u = \prod_{i=1}^{N} x_i^{\alpha_i} \tag{8}$$

Indirect Utility:

$$v = m \prod_{i=1}^{N} \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} \tag{9}$$

Expenditure Function:

$$e = u \prod_{i=1}^{N} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} \tag{10}$$

Price Index:

$$p_u = \prod_{i=1}^{N} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} \tag{11}$$

Typical Demand Equation:

$$x_i = \frac{\alpha_i m}{p_i} \tag{12}$$

Uncompensated Demand Elasticity:

$$\eta_i = 1 \tag{13}$$

Income Elasticity:

$$\eta_m = 1 \tag{14}$$

Expenditure Share:

$$s_i = \alpha_i \tag{15}$$

1.3 Constant Elasticity of Substitution

Elasticity of substitution can vary; homothetic.

Utility:

$$u = \left(\sum_{i=1}^{N} \alpha_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{16}$$

Indirect Utility Function:

$$v = m \left(\sum_{i=1}^{N} \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{\sigma-1}} \tag{17}$$

Expenditure Function:

$$e = u \left(\sum_{i=1}^{N} \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{18}$$

Price Index:

$$p_u = \left(\sum_{i=1}^{N} \alpha_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{19}$$

Typical Demand Equation:

$$x_i = \frac{\alpha_i m}{p_i^{\sigma}} \left(\sum_{i=1}^{N} \alpha_i p_i^{1-\sigma} \right)^{-1} = \frac{\alpha_i m}{p_u} \left(\frac{p_u}{p_i} \right)^{\sigma} \tag{20}$$

Uncompensated Demand Elasticity:

$$\eta_i = -(1 - s_i)\sigma - s_i \tag{21}$$

Income Elasticity:

$$\eta_m = 1 \tag{22}$$

Expenditure Share:

$$s_i = \alpha_i \left(\frac{p_u}{p_i} \right)^{\sigma-1} \tag{23}$$

1.4 Linear Expenditure System

Does not impose homotheticity.

Utility Function:

$$u = \prod_i (x_i - \gamma_i)^{\alpha_i} \tag{24}$$

Supernumary Income (excess of income over required expenditure):

$$m_{sn} = m - \sum_i p_i \gamma_i \tag{25}$$

Indirect Utility Function:

$$v = m_{sn} \prod_i \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} \tag{26}$$

Expenditure Function:

$$e = \sum_i p_i \gamma_i + u \prod_i \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} \tag{27}$$

Typical Demand Equation:

$$x_i = \gamma_i + \frac{\alpha_i m_{sn}}{p_i} \tag{28}$$

Expenditure Share:

$$s_i = \frac{p_i \gamma_i}{m} + \alpha_i \frac{m_{sn}}{m} \tag{29}$$

1.5 Transcendental Logarithmic Indirect Utility

Does not impose homotheticity.

Indirect Utility Function:

$$\ln v = \sum_i \alpha_i \ln \frac{p_i}{m} + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln \frac{p_i}{m} \ln \frac{p_j}{m}$$

Typical Expenditure Share:

$$\omega_i = \frac{\alpha_i + \sum_j \beta_{ij} \ln \frac{p_j}{m}}{\sum_k \left(\alpha_k + \sum_j \beta_{kj} \ln \frac{p_j}{m} \right)} \tag{30}$$

1.6 Generalized Leontief

Under construction...

2 Intertemporal Utility

The following assume an intertemporal budget constraint of the form below, where B_0 is the initial stock of financial assets and y is total income from all other sources:

$$\int_0^\infty p(s)c(s)e^{-rs}ds = B_0 + \int_0^\infty y(s)e^{-rs}ds \tag{31}$$

For convenience, let W be total wealth:

$$W = B_0 + \int_0^\infty y(s)e^{-rs}ds$$

2.1 Logarithmic

Unitary intertemporal elasticity of substitution.

Utility Function:

$$U = \int_0^\infty \ln(c(s))e^{-\rho s}ds \tag{32}$$

First-order condition for consumption at time s:

$$\frac{1}{c(s)}e^{-\rho s} = \Lambda p(s)e^{-rs} \tag{33}$$

Consumption expenditure at time s in terms of expenditure at time 0:

$$p(s)c(s) = p(0)c(0)e^{(r-\rho)s} \tag{34}$$

Initial expenditure as a function of wealth:

$$p(0)c(0) = \rho W$$

2.2 Constant Intertemporal Elasticity of Substitution

Utility Function:

$$U = \int_0^\infty c(s)^{\frac{\sigma-1}{\sigma}}e^{-\rho s}ds \tag{35}$$

First-order condition for consumption at time s:

$$\left(\frac{\sigma - 1}{\sigma}\right) c(s)^{-1/\sigma} e^{-\rho s} = \Lambda p(s) e^{-rs} \tag{36}$$

Consumption expenditure at time s in terms of expenditure at time 0:

$$p(s)c(s) = p(0)c(0) \left(\frac{p(s)}{p(0)}\right)^{1-\sigma} e^{\sigma(r-\rho)s} \tag{37}$$

Initial expenditure as a function of wealth:

$$p(0)c(0) = \frac{W}{\Gamma}, \quad \Gamma = \int_0^\infty \left(\frac{p(s)e^{-rs}}{p(0)}\right)^{1-\sigma} e^{-\sigma\rho s} ds \tag{38}$$

2.3 Intertemporal Analog to the Linear Expenditure System

Additional parameters allow consumption to be hump-shaped over the life cycle.

Utility Function:

$$U = \sum_{s=0}^{T} \frac{(c_s - \gamma_s)^{\alpha_s}}{(1+\rho)^s} \tag{39}$$

3 Production

In the following, w_i is the price of input i and p is the price of output.

3.1 Leontief

Production Function:

$$q = \min\left\{\frac{x_i}{\alpha_i}\right\} \tag{40}$$

Cost Function:

$$C = q\sum_i \alpha_i w_i \tag{41}$$

Unit Cost Function:

$$c = \sum_i \alpha_i w_i \tag{42}$$

Factor Demand Equation:

$$x_i = \alpha_i q \tag{43}$$

3.2 Cobb-Douglas

Production Function:

$$q = A\prod_i x_i^{\alpha_i} \tag{44}$$

Cost Function:

$$C = \frac{1}{A}\prod_i \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i} q \tag{45}$$

Unit Cost Function:

$$c = \frac{1}{A}\prod_i \left(\frac{w_i}{\alpha_i}\right)^{\alpha_i} \tag{46}$$

Factor Demand Equation:

$$x_i = \frac{\alpha_i q c}{w_i} \tag{47}$$

3.3 Constant Elasticity of Substitution

Production Function:

$$q = A \left(\sum_i \delta_i^{\frac{1}{\sigma}} x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{48}$$

Cost Function:

$$C = \frac{q}{A} \left(\sum_i \delta_i w_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{49}$$

Unit Cost Function:

$$c = \frac{1}{A} \left(\sum_i \delta_i w_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{50}$$

Factor Demand Equation:

$$x_i = \delta_i A^{\sigma-1} \left(\frac{c}{w_i} \right)^{\sigma} q \tag{51}$$

Cost Share:

$$s_i = \delta_i \left(\frac{Ac}{w_j} \right)^{\sigma-1} \tag{52}$$

3.4 Transcendental Logarithmic Cost Function

Unit Cost Function:

$$\ln c = \alpha_0 + \sum_i \alpha_i \ln w_i + \frac{1}{2} \sum_i \sum_j \beta_{ij} \ln w_i \ln w_j \tag{53}$$

Cost Share:

$$s_i = \alpha_i + \sum_j \beta_{ij} \ln w_j \tag{54}$$